21-241 Exam 1
Fall 2024

Name: _____

andrewID: _____

Directions: Thoroughly read the following.

Before The Exam:

- Do not open the exam until instructed to do so.
- Clearly print your name and andrewID in the designated locations above.
- Place your Carnegie Mellon ID on your desk in front of you.
- Turn off your electronic devices and store them, along with your notes and books, out of sight (e.g. in a closed bag)

During The Exam:

- Your work must be shown clearly and legibly with appropriate justifications (using a pencil or eraseable pen is always a good idea).
- Write your solutions only on the pages assigned to the appropriate questions. You may use the extra page(s) at the back of the exam for scratch work or continuing solutions; if you continue your solutions on the extra page(s), you should clearly indicate the page number where the solution is continued so the grader will see it. Do not remove any pages from the staples
- You may not use notes, books, other reference materials, or electronic devices.
- If you need to leave the classroom (e.g. to use the restroom), please raise your hand, turn in your CMU ID card to a proctor, and leave your electronic devices in the classroom.

After The Exam

- Stop working immediately when you are instructed to do so.
- You must hand in your own exam and you must hand in all pages of your exam.

1. Determine whether the following statements are true or false. If a statement is true, prove it. If it is false, explain why or provide a counterexample.

   (a) If A is invertible and $AB = A$, then $B = I$.

   (b) Let A be an $m \times n$ matrix. If $C(A) = \mathbb{R}^m$ and $N(A) = \{\mathbf{0}\}$, then $m = n$.

   (c) The subspace $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix} \right\}$ of $\mathbb{R}^3$ has dimension 2.

2. Note that this problem spans two pages. Suppose a matrix A has RREF $\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

   Answer the following questions and explain your reasoning.

   (a) Does $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b} \in \mathbb{R}^3$?

   (b) Are the columns of A linearly independent?

(c) What are the dimensions of the four fundamental subspaces?

(d) Find a basis for the nullspace $N(A)$.

3. Note that this problem spans two pages. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

(a) Reduce A to RREF. Be sure to explain your steps.

(b) Find a basis for the column space $C(A)$. Explain your reasoning.

(c) What does $C(A)$ look like, geometrically? Explain your reasoning.

(d) Is $\mathbf{b} \in C(A)$? If so, express it as a linear combination of the basis vectors of $C(A)$. If not, explain why.

4. Note that this problem spans two pages. Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 2y \\ 0 \end{bmatrix}$$

(a) Show that T is a linear transformation using the definition of linear transformations.

(b) Find the matrix associated to T. Explain your reasoning.

(c) Find a basis for the image (i.e. range) of T.

(d) Find the matrix associated to T^{-1}, if it exists. If it does not exist, explain why.

5. Suppose $\{\mathbf{v}, \mathbf{w}\} \subset \mathbb{R}^n$ is linearly independent. Use the definition of linear independence to show that $\{\mathbf{v}, a\mathbf{w} + b\mathbf{v}\}$ is linearly independent if $a \neq 0$.

This page is for scratch work or continued solutions